Bianchi I model in terms of nonstandard loop quantum cosmology: Quantum dynamics.

Przemysław Małkiewicz†, Włodzimierz Piechocki‡, and Piotr Dzierżak§
Theoretical Physics Department, Institute for Nuclear Studies
Hoża 69, 00-681 Warsaw, Poland;
†pmalk@fuw.edu.pl, ‡piech@fuw.edu.pl, §pdzi@fuw.edu.pl

(Dated: March 3, 2011)

Abstract

We analyze the quantum Bianchi I model in the setting of the nonstandard loop quantum cosmology. Elementary observables are used to quantize the volume operator. The spectrum of the volume operator is bounded from below and discrete. The discreteness may imply a foamy structure of spacetime at semiclassical level. The results are described in terms of a free parameter specifying loop geometry to be determined in astro-cosmo observations. An evolution of the quantum model is generated by the so-called true Hamiltonian, which enables an introduction of a time parameter valued in the set of all real numbers.

PACS numbers: 98.80.Qc, 04.60.Pp, 04.20.Jb

I. INTRODUCTION

The great challenge is quantization of the Belinskii-Khalatnikov-Lifshitz (BKL) theory [1–3]. It is a generic solution of General Relativity (GR) that does not rely on any symmetry conditions. It presents an evolution of the universe near the space-like cosmological singularity (CS) with diverging gravitational and matter fields invariants. The singular solution may be applied both to the future singularity (Big Crunch) and past singularity (Big Bang). The BKL scheme also appears in the low energy limit of superstring models, where it is linked to the hyperbolic Kac-Moody algebras [4].

The dynamics of the BKL model, close to the singularity, may be approximated by the two key ingredients: (i) Kasner-type evolutions well approximated by the Bianchi I model, and (ii) spike-type transitions described by the Bianchi II model. The former case occurs when time derivatives in the equations of motion are important, whereas the latter case appears when space-like gradients play the crucial role. The general BKL type evolution, near the cosmological singularity, consists of a sequence of epochs (i) and (ii), which may lead to the oscillatory and chaotic type dynamics.

Present paper concerns quantization of the Bianchi I model with massless scalar field. It is a companion paper to our recent paper [5], presenting classical dynamics of the Bianchi I model in terms of loop geometry.

Through the paper we apply the reduced phase space, RPS, quantization method developed by us recently [6–10]. It is an alterative method to the Dirac quantization recently applied to the quantization of the Bianchi I model [11–16].

In section II, we recall some elements of the classical formalism [5] for self-consistency. We redefine an evolution parameter and elementary observables, and introduce the true Hamiltonian. Section III is devoted to the quantization of the classical model. Compound observables are quantized in terms of elementary observables. We present solution to the eigenvalue problem for the volume operators. Examination of an evolution of the quantum system completes this section. We conclude in the last section. Unitarily non-equivalent representations of the volume operators are briefly discussed in the appendix A. We make comments on the operators ordering problem in the appendix B.

II. PREPARATIONS TO QUANTIZATION

The Bianchi I model with massless scalar field is described by the line element

$$ds^2 = -N^2\, dt^2 + \sum_{i=1}^{3} a_i^2(t)\, dx_i^2, \tag{1}$$

where

$$a_i(\tau) = a_i(0)\left(\frac{\tau}{\tau_0}\right)^{\mathrm{k}_i}, \qquad d\tau = N\, dt, \qquad \sum_{i=1}^{3} \mathrm{k_i} = 1 = \sum_{i=1}^{3} \mathrm{k_i^2} + \mathrm{k}_\phi^2, \tag{2}$$

and where k_ϕ describes matter field density ($\mathrm{k}_\phi = 0$ corresponds to the Kasner model). For a clear exposition of the *classical* singularity aspects of the Bianchi I model, in terms of the loop geometry, we recommend [12].

A. Hamiltonian constraint

The gravitational part of the classical Hamiltonian, for the Bianchi I model with massless scalar field, reads [5]

$$H_g = -\gamma^{-2} \int_{\mathcal{V}} d^3x \; N e^{-1} \varepsilon_{ijk} E^{aj} E^{bk} F^i_{ab} \,, \tag{3}$$

where γ is the Barbero-Immirzi parameter, $\mathcal{V} \subset \Sigma$ is the fiducial volume, Σ is spacelike hypersurface, N denotes the lapse function, ε_{ijk} is the alternating tensor, E^a_i is a densitized vector field, $e := \sqrt{|\det E|}$, and where F^i_{ab} is the curvature of an $SU(2)$ connection A^i_a.

The resolution of the singularity, obtained within LQC, is based on rewriting the curvature F^k_{ab} in terms of the holonomy around a loop by making use of the formula [5]

$$F^k_{ab} = -2 \lim_{Ar\,\Box_{ij} \rightarrow 0} Tr \left(\frac{h_{\Box_{ij}} - 1}{Ar\,\Box_{ij}} \right) \tau^k \, {}^o\omega^i_a \, {}^o\omega^j_a, \tag{4}$$

where

$$h_{\Box_{ij}} = h^{(\mu_i)}_i h^{(\mu_j)}_j (h^{(\mu_i)}_i)^{-1} (h^{(\mu_j)}_j)^{-1} \tag{5}$$

is the holonomy of the gravitational connection around the square loop $\Box_{ij}$. The loop is taken over a face of an elementary cell, each of whose sides has length $\mu_j L_j$ with respect to the flat *fiducial* metric ${}^o q_{ab} := \delta_{ij} \, {}^o\omega^i_a \, {}^o\omega^j_a$; the fiducial triad ${}^o e^a_k$ and cotriad ${}^o\omega^k_a$ satisfy ${}^o\omega^i_a \, {}^o e^a_j = \delta^i_j$; $Ar\,\Box_{ij}$ denotes the area of the square $\Box_{ij}$; $L_1 L_2 L_3 = V_0$, where $V_0 = \int_{\mathcal{V}} \sqrt{{}^o q} d^3x$ is the fiducial volume of $\mathcal{V}$ with respect to the fiducial metric.

The holonomy in the fundamental, $j = 1/2$, representation of $SU(2)$ reads

$$h^{(\mu_i)}_i = \cos(\mu_i c_i/2) \, \mathbb{I} + 2 \sin(\mu_i c_i/2) \, \tau_i, \tag{6}$$

where $\tau_i = -i\sigma_i/2$ (σ_i are the Pauli spin matrices). The connection A^k_a and the density weighted triad E^a_k are determined by the conjugate variables c_k and p_k as follows

$$A^i_a = c_i \, L_i^{-1} \, {}^o\omega^i_a, \qquad E^a_i = p_i \, L_j^{-1} \, L_k^{-1} \, {}^o e^a_i, \tag{7}$$

where

$$c_i = \gamma \, \dot{a}_i \, L_i, \qquad |p_i| = a_j \, a_k \, L_j \, L_k, \tag{8}$$

(the dot over a_i denotes derivative with respect to the cosmological time), and where

$$\{c_i, p_j\} = 8\pi G \gamma \delta_{ij} \tag{9}$$

Making use of (3) and (4) leads to H_g in the form [5]

$$H_g = \lim_{\mu_1, \mu_2, \mu_3 \rightarrow 0} H^{(\mu_1 \, \mu_2 \, \mu_3)}_g, \tag{10}$$

where

$$H^{(\mu_1 \, \mu_2 \, \mu_3)}_g = -\frac{\mathrm{sign}(p_1 p_2 p_3)}{2\pi G \gamma^3 \mu_1 \mu_2 \mu_3} \sum_{ijk} N \, \varepsilon^{ijk} \, Tr\Big(h^{(\mu_i)}_i h^{(\mu_j)}_j (h^{(\mu_i)}_i)^{-1} (h^{(\mu_j)}_j)^{-1} h^{(\mu_k)}_k \{ (h^{(\mu_k)}_k)^{-1}, V \} \Big), \tag{11}$$

and where $V = a_1 a_2 a_3 V_0$ is the volume of the elementary cell $\mathcal{V}$ with respect to the *physical* metric $q_{ab} = a_1 a_2 a_3 \, {}^o q_{ab}$.

The total Hamiltonian for Bianchi I universe with a massless scalar field, ϕ, reads

$$H = H_g + H_\phi \approx 0, \tag{12}$$

where H_g is defined by (10). The Hamiltonian of the scalar field reads $H_\phi = N\, p_\phi^2 |p_1 p_2 p_3|^{-\frac{1}{2}}/2$, where ϕ and p_ϕ are the elementary variables satisfying $\{\phi, p_\phi\} = 1$. The relation $H \approx 0$ defines the constraint on phase space of considered gravitational system.

Making use of (6) we calculate (11) and get the *modified* total Hamiltonian $H_g^{(\lambda)}$ corresponding to (12) in the form

$$H^{(\lambda)}/N = -\frac{1}{8\pi G\gamma^2}\,\frac{\mathrm{sgn}(p_1p_2p_3)}{\mu_1\mu_2\mu_3}\left[\sin(c_1\mu_1)\sin(c_2\mu_2)\,\mu_3\,\mathrm{sgn}(p_3)\sqrt{\frac{|p_1p_2|}{|p_3|}} + \mathrm{cyclic}\right] + \frac{p_\phi^2}{2\,V}, \tag{13}$$

where $V = \sqrt{|p_1p_2p_3|}$. In what follows we assume that

$$\mu_k := \sqrt{\frac{1}{|p_k|}}\,\lambda, \tag{14}$$

where λ is a free *parameter* of our model.

The choice (14) for μ_k leads, in the Dirac quantization [13, 14, 16], to the dependance of dynamics on $\mathcal{V} \subset \Sigma$. In what follows, we specialize our considerations to the case $\mathcal{V} = \Sigma = \mathbb{T}^3$. In such a case the volume *observable*, $V = \int_{\mathcal{V}} \sqrt{q}\, d^3x$ (where g denotes the determinant of the physical metric q_{ab} on Σ), characterizes the entire space part of the universe. Thus, $\mathcal{V}$ is chosen unambiguously and V is physical. In the case when one considers the Bianchi I model with the $\mathbb{R}^3$ topology, the volume $\mathcal{V} \subset \Sigma = \mathbb{R}^3$ is only an auxiliary *tool* devoid of any physical meaning[1].

The present paper is a quantum version of our recent paper [5], where we consider the Bianchi I model with the $\mathbb{T}^3$ topology. The aim of both our papers is presenting a quantum Bianchi I model in terms of the *nonstandard* LQC, which is an alternative to the *standard* LQC results [14, 15] (with $\mathcal{V} = \mathbb{T}^3$ and the choice (14)). The results obtained within these two methods are similar. Detailed comparison is beyond the scope of the present paper, but will be presented elsewhere after we complete quantization of the Bianchi II model [18].

Let us analyze the dependance of our results on the choice of coordinates in Σ, and consequently on the choice of the fiducial volume $V_0 = L_1L_2L_3$. It is clear that L_k is a *coordinate* length, whereas a_kL_k is the *physical* one. The latter is invariant with respect to a change of the system of coordinates so we have $a_kL_k = a'_kL'_k$, while $L_k \to L'_k$. Since, due to (8), we have

$$c_i = \gamma\frac{\dot{a}_i}{a_i}\, a_iL_i, \qquad |p_i| = a_jL_j\, a_kL_k, \tag{15}$$

the canonical variables c_i and p_i do not depend on the choice of the coordinate system. Thus, the destination variables, defined by Eq. (20), share this property too. The holonomy

[1] It is tempting to introduce the notion of a *local* volume, i.e. a sort of 'density' of the physical volume operator V. It appears that the local volume may be defined in the case of *any* topology of space, however it would depend on the choice of coordinates on Σ. Similar notion has been used by Martin Bojowald (see, Appendix in [17]), while considering lattice refining in loop quantum cosmology.

variable $h_k^{(\mu_k)}$ depends on $\mu_k c_k$, and with our choice (14) leads to

$$\mu_k c_k = \lambda \frac{c_k}{\sqrt{|p_k|}} = \lambda' \frac{c_k'}{\sqrt{|p_k'|}} = \mu_k' c_k', \tag{16}$$

which proves that the holonomy variable does not depend on the choice of coordinates[2]. The flux variable is p_k so it does not depend on the choice of coordinates either. Since holonomy and flux are basic variables of the formalism, our final results do not depend on the choice of coordinates (the choice of V_0). This is why the variables β_k and v_k share this property as well. Our results do depend on $\mathcal{V}$, but it is correct since $\mathcal{V} = \mathbb{T}^3$ is the whole space.

We wish to emphasize that (13) is not an *effective* Hamiltonian for quantum dynamics [13], but a *classical* Hamiltonian *modified* by approximating the curvature F^k_{ab} by holonomy of connection around a loop with *finite* length. Our approach is quite different from the so-called polymerization method where the replacement $c \to \sin(c\mu)/\mu$ in the Hamiltonian is treated as some kind of an effective *quantization*. Our method has been presented with all details and compared with the Dirac quantization method in [6, 7, 9, 10]. For an extended *motivation* of our approach we recommend an appendix of [9].

In the gauge $N = V = \sqrt{|p_1 p_2 p_3|}$, the Hamiltonian modified by loop geometry reads

$$H^{(\lambda)} = -\frac{1}{8\pi G\gamma^2\lambda^2}\left[\mathrm{sgn}(p_1p_2)|p_1p_2|^{3/2}\sin\big(\lambda\frac{c_1}{\sqrt{|p_1|}}\big)\sin\big(\lambda\frac{c_2}{\sqrt{|p_2|}}\big) + \mathrm{cyclic}\right] + \frac{p_\phi^2}{2}. \tag{17}$$

Since we consider the *relative* dynamics [5, 7] our results, in what follows, are gauge independent.

Equation (17) corresponds to the effective quantization of the standard LQC. In the nonstandard LQC Eq. (17) is treated as the constraint, which is to be imposed into the *classical* dynamics. In the standard LQC one implements this constraint into an operator constraint defining *quantum* dynamics (kernel of this operator is used to find the physical Hilbert space). Thus, in the reduced phase space quantization (nonstandard LQC) there is no quantum Hamiltonian constraint, contrary to the Dirac quantization, i.e. the standard LQC (which is motivated from LQG). In both cases one *modifies*, to some extent, gravity theory: already at the classical level in the reduced phase space quantization, only at the quantum level in the Dirac quantization.

The Poisson bracket is defined to be

$$\{\cdot,\cdot\} := 8\pi G\gamma \sum_{k=1}^{3}\left[\frac{\partial\cdot}{\partial c_k}\frac{\partial\cdot}{\partial p_k} - \frac{\partial\cdot}{\partial p_k}\frac{\partial\cdot}{\partial c_k}\right] + \frac{\partial\cdot}{\partial\phi}\frac{\partial\cdot}{\partial p_\phi} - \frac{\partial\cdot}{\partial p_\phi}\frac{\partial\cdot}{\partial\phi}, \tag{18}$$

where $(c_1, c_2, c_3, p_1, p_2, p_3, \phi, p_\phi)$ are canonical variables. The dynamics of a function ξ on a phase space is defined by

$$\dot{\xi} := \{\xi, H^{(\lambda)}\}, \qquad \xi \in \{c_1, c_2, c_3, p_1, p_2, p_3, \phi, p_\phi\}. \tag{19}$$

The dynamics is defined by the solutions to (19) satisfying the constraint $H^{(\lambda)} \approx 0$. The solutions of (19) ignoring the constraint are *nonphysical*.

[2] We have $\lambda = \lambda'$ since λ is a *physical* length.

In what follows we shift from (c_k, p_k) to another canonical variables (v_k, β_k)

$$\beta_k := \frac{c_k}{\sqrt{|p_k|}}, \qquad v_k := |p_k|^{3/2}, \tag{20}$$

(where $k = 1, 2, 3$) since they are proper variables to examine the singularity aspects of our model [5]. In this paper we restrict our considerations to $v_k \geq 0$, since we wish to ascribe to it a directional *volume* observable.

B. Correspondence with FRW observables

We redefine the original elementary Bianchi observables that has been found in [5] as follows

$$\mathrm{O}_i = \frac{1}{3\kappa\gamma} \frac{v_i \sin(\lambda\beta_i)}{\lambda}, \tag{21}$$

and

$$\mathrm{A}_i = \frac{1}{3\kappa} \ln\left(\frac{\left|\tan\left(\frac{\lambda\beta_i}{2}\right)\right|}{\frac{\lambda}{2}}\right) + \frac{3}{2\sqrt{3}} \frac{\mathrm{sgn}(p_\phi)\big(\mathrm{O}_j + \mathrm{O}_k\big)\,\phi}{\sqrt{\mathrm{O}_1\mathrm{O}_2 + \mathrm{O}_1\mathrm{O}_3 + \mathrm{O}_2\mathrm{O}_3}}, \tag{22}$$

where $\kappa^2 := 4\pi G/3$. Dropping subscripts leads to the elementary observables of the FRW model found in [9]. One may verify that the algebra of redefined observables is

$$\{\mathrm{O}_i, \mathrm{O}_j\} = 0, \qquad \{\mathrm{A}_i, \mathrm{O}_j\} = \delta_{ij}, \qquad \{\mathrm{A}_i, \mathrm{A}_j\} = 0. \tag{23}$$

Our main concern is quantization of the *volume* observable defined as follows [5]

$$V = (v_1 v_2 v_3)^{1/3}, \tag{24}$$

where

$$v_i = 3\kappa\gamma\lambda|\mathrm{O}_i|\, \cosh\left(\frac{3\sqrt{\pi G}\,\big(\mathrm{O}_j + \mathrm{O}_k\big)\,\phi}{\sqrt{\mathrm{O}_1\mathrm{O}_2 + \mathrm{O}_1\mathrm{O}_3 + \mathrm{O}_2\mathrm{O}_3}} - 3\kappa\mathrm{A}_i\right). \tag{25}$$

In an 'isotropic' case (i=j=k) we get the expression for the volume observable of the FRW model [9].

C. Redefinitions of evolution parameter

Since the observables $\mathrm{O_i}$ are constants of motion in ϕ and $\phi \in \mathbb{R}$, it is possible to make the following redefinition of an evolution parameter

$$\varphi := \frac{\sqrt{3}\,\phi}{2\sqrt{\mathrm{O}_1\mathrm{O}_2 + \mathrm{O}_1\mathrm{O}_3 + \mathrm{O}_2\mathrm{O}_3}} \tag{26}$$

so we have

$$v_i = 3\kappa\gamma\lambda|\mathrm{O}_i|\, \cosh 3\kappa\big((\mathrm{O}_j + \mathrm{O}_k)\,\varphi - \mathrm{A}_i\big), \tag{27}$$

which simplifies further considerations.

D. Redefinitions of elementary observables

One can make the following redefinitions

$$\mathcal{A}_i := \mathrm{A}_i - (\mathrm{O}_j + \mathrm{O}_k)\,\varphi. \tag{28}$$

Thus, the directional volume (27) becomes

$$v_i := |w_i|, \qquad w_i = 3\kappa\gamma\lambda O_i \cosh(3\kappa\mathcal{A}_i). \tag{29}$$

The algebra of observables reads

$$\{\mathrm{O}_i, \mathrm{O}_j\} = 0, \quad \{\mathcal{A}_i, \mathrm{O}_j\} = \delta_{ij}, \quad \{\mathcal{A}_i, \mathcal{A}_j\} = 0, \tag{30}$$

where the Poisson bracket is defined to be

$$\{\cdot,\cdot\} := \sum_{k=1}^{3}\Big(\frac{\partial\cdot}{\partial\mathcal{A}_k}\frac{\partial\cdot}{\partial O_k} - \frac{\partial\cdot}{\partial O_k}\frac{\partial\cdot}{\partial\mathcal{A}_k}\Big). \tag{31}$$

E. Structure of phase space

All considerations carried out in the previous section have been done under the assumption that the observables O_1, O_2 and O_3 have no restrictions. The inspection of (22), (25) and (28) shows that the domain of definition of the elementary observables reads

$$D := \{(\mathcal{A}_k, \mathrm{O}_k)\,|\,\mathcal{A}_k \in \mathbb{R}, \quad \mathrm{O}_1\mathrm{O}_2 + \mathrm{O}_1\mathrm{O}_3 + \mathrm{O}_2\mathrm{O}_3 > 0\}, \tag{32}$$

where $k = 1,2,3$. The restriction $\mathrm{O}_1\mathrm{O}_2 + \mathrm{O}_1\mathrm{O}_3 + \mathrm{O}_2\mathrm{O}_3 > 0$ is a consequence of the Hamiltonian constraint (see, [5] for more details).

In what follows we consider two cases:

1. Kasner-unlike dynamics: (a) $\mathrm{O}_i > 0$, $\mathrm{O}_j > 0$, $\mathrm{O}_k > 0$, which describes all three directions expanding (b) $\mathrm{O}_i < 0$, $\mathrm{O}_j < 0$, $\mathrm{O}_k < 0$, with all directions shrinking.

2. Kasner-like dynamics: (a) $\mathrm{O}_i > 0$, $\mathrm{O}_j > 0$, $\mathrm{O}_k < 0$, which describes two directions expanding and one direction shrinking; (b) $\mathrm{O}_i < 0$, $\mathrm{O}_j < 0$, $\mathrm{O}_k > 0$, with two directions shrinking and one expanding.

This classification presents all possible nontrivial cases. Our terminology fits the one used in [13] due to the relation $\mathrm{O}_i = 6\kappa k_i K$, $(0 < K = const)$, where constants k_i are defined by (2).

F. True Hamiltonian

Now, we define a generator of an evolution called a true Hamiltonian $\mathbb{H}$. Making use of (28), and $\mathrm{O}_i = const$ (see [5]), we get

$$\{\mathcal{A}_i, \mathbb{H}\} := \frac{d\mathcal{A}_i}{d\varphi} = -(\mathrm{O}_j + \mathrm{O}_k), \quad \{\mathrm{O}_i, \mathbb{H}\} := \frac{d\mathrm{O}_i}{d\varphi} = 0. \tag{33}$$

The solution to (33) is easily found to be

$$\mathbb{H} = \mathrm{O}_1\mathrm{O}_2 + \mathrm{O}_1\mathrm{O}_3 + \mathrm{O}_2\mathrm{O}_3. \tag{34}$$

The true Hamiltonian is defined on the *reduced* phase space which is devoid of constraints.

III. QUANTIZATION

A. Representation of elementary observables

We use the Schrödinger representation for the algebra (30) defined as

$$\mathrm{O}_k \to \widehat{\mathrm{O}}_k\, f_k(x_k) := \frac{\hbar}{i}\frac{d}{dx_k}\, f_k(x_k), \quad \mathcal{A}_k \to \widehat{\mathcal{A}}_k\, f_k(x_k) := x_k\, f_k(x_k), \quad k = 1,2,3. \tag{35}$$

One may verify that

$$[\widehat{\mathrm{O}}_i, \widehat{\mathrm{O}}_j] = 0, \quad [\widehat{\mathcal{A}}_i, \widehat{\mathcal{A}}_j] = 0, \quad [\widehat{\mathcal{A}}_i, \widehat{\mathrm{O}}_j] = i\hbar\,\delta_{ij}. \tag{36}$$

The representation is defined formally on some dense subspaces of a Hilbert space to be specified later.

B. Kasner-unlike case

The condition $\mathrm{O}_1\mathrm{O}_2 + \mathrm{O}_1\mathrm{O}_3 + \mathrm{O}_2\mathrm{O}_3 > 0$ is automatically satisfied in this case, because $\mathrm{O}_1, \mathrm{O}_2$ and O_3 are of the same sign. To be specific, let us consider (1a); the case (1b) can be done by analogy.

Let us quantize the directional volumes by means of w_i defined in (29). A standard procedure gives[3]

$$\hat{w} := \frac{3\kappa\gamma\lambda}{2}\Big(\widehat{\mathrm{O}}\,\cosh\big(3\kappa\widehat{\mathcal{A}}\big) + \cosh\big(3\kappa\widehat{\mathcal{A}}\big)\,\widehat{\mathrm{O}}\Big) = -\frac{3ia}{2}\Big(2\cosh(bx)\frac{d}{dx} + b\sinh(bx)\Big), \tag{37}$$

where $a := \kappa\gamma\lambda\hbar$ and $b := 3\kappa$, and where we have used the representation for the elementary observables defined by (35).

In what follows we solve the eigenvalue problem for the operator $\hat{w}$ and identify its domain of self-adjointness.

Let us consider the invertible mapping $L^2(\mathbb{R}, dx) \ni \psi \to \tilde{U}\psi =: f \in L^2(\mathbb{I}, dy)$ defined by

$$\tilde{U}\psi(x) := \frac{\psi(\ln|\mathrm{tg}^{1/b}(\frac{by}{2})|)}{\sin^{1/2}(by)} =: f(y), \quad x \in \mathbb{R}, \quad y \in \mathbb{I} := (0, \pi/b). \tag{38}$$

We have

$$\begin{aligned} \langle\psi|\psi\rangle &= \int_{-\infty}^{\infty} \overline{\psi}\psi\, dx \\ &= \int_0^{\frac{\pi}{b}} \overline{\psi}(\ln|\mathrm{tg}^{1/b}(\frac{by}{2})|)\psi(\ln|\mathrm{tg}^{1/b}(\frac{by}{2})|)d\big(\ln|\mathrm{tg}^{1/b}(\frac{by}{2})|\big) \\ &= \int_0^{\frac{\pi}{b}} \overline{\psi}(\ln|\mathrm{tg}^{1/b}(\frac{by}{2})|)\psi(\ln|\mathrm{tg}^{1/b}(\frac{by}{2})|)\frac{dy}{\sin(by)} \\ &= \int_0^{\frac{\pi}{b}} \overline{\frac{\psi(\ln|\mathrm{tg}^{1/b}(\frac{by}{2})|)}{\sin^{1/2}(by)}}\,\frac{\psi(\ln|\mathrm{tg}^{1/b}(\frac{by}{2})|)}{\sin^{1/2}(by)}\, dy = \langle\tilde{U}\psi|\tilde{U}\psi\rangle. \end{aligned} \tag{39}$$

[3] In what follows we drop subscripts of observables to simplify notation.

Thus, the mapping (38) is isometric and hence *unitary*.

Now, let us see how the operator $\hat{w}$ transforms under the unitary map (38). The transformation consists of the change of an independent variable

$$x \mapsto y := \frac{2}{b}\arctan(e^{bx}), \tag{40}$$

which leads to

$$-\frac{ia}{2}\Big(2\cosh(bx)\frac{d}{dx} + b\sinh(bx)\Big) \mapsto -ia\frac{d}{dy} + i\frac{ab}{2}\cot(by), \tag{41}$$

and re-scaling with respect to a dependent variable

$$-ia\frac{d}{dy} + i\frac{ab}{2}\cot(by) \mapsto \sin^{-1/2}(by)\Big(-ia\frac{d}{dy} + i\frac{ab}{2}\cot(by)\Big)\sin^{1/2}(by) = -ia\frac{d}{dy}. \tag{42}$$

In the process of mapping

$$\hat{w} \mapsto \tilde{U}\,\hat{w}\,\tilde{U}^{-1} = -ia\frac{d}{dy} =: \breve{w}, \tag{43}$$

we have used two identities: $\sin(by) = 1/\cosh(bx)$ and $\sinh(bx) = -\cot(by)$.

Since $w > 0$ (for $\mathrm{O} > 0$), we *assume* that the spectrum of $\breve{w}$ consists of positive eigenvalues. To implement this assumption, we define $\breve{w} := \sqrt{\breve{w}^2}$ and consider the eigenvalue problem

$$-a^2\frac{d^2}{dy^2}f_\nu = \nu^2 f_\nu, \qquad y \in (0, \pi/b). \tag{44}$$

There are two independent solutions for each value of ν^2 (where $\nu \in \mathbb{R}$), namely: $\sin(\frac{\nu}{a}y)$ and $\cos(\frac{\nu}{a}y)$. Removing this degeneracy leads to required positive eigenvalues of $\breve{w}$. We achieve that in a standard way by requiring that the eigenvectors vanish at the boundaries, i.e, at $y = 0$ and $y = \pi/b$. As the result we get the following spectrum

$$f_\nu = N\sin(\frac{\nu}{a}y), \qquad \nu^2 = (nab)^2, \quad n = 0, 1, 2, \dots \tag{45}$$

It should be noted that for $n = 0$, the eigenvector is a null state and thus the lowest eigenvalue is $\nu^2 = (ab)^2$. Next, we define the Hilbert space to be the closure of the span of the eigenvectors (45). The operator $\breve{w}^2 = -a^2\frac{d^2}{dy^2}$ is essentially self-adjoint on this span by the construction. Due to the spectral theorem [19] we may define an essentially self-adjoint operator $\breve{w} = \sqrt{-a^2\frac{d^2}{dy^2}}$ as follows

$$\breve{w}f_\nu := \nu f_\nu, \qquad \nu = ab,\, 2ab,\, 3ab, \dots \tag{46}$$

We have considered the case $w > 0$. The case $w < 0$ does not require changing of the Hilbert space. The replacement $\hat{w} \mapsto -\hat{w}$ leads to $\nu \mapsto -\nu$.

Finally, we find that the inverse mapping from $L^2(\mathbb{I}, dy)$ to $L^2(\mathbb{R}, dx)$ for the eigenvectors of $\breve{w}$ yields

$$\sin\big(\frac{\nu}{a}y\big) = f_\nu(y) \mapsto \tilde{U}^{-1}f_\nu(y) := \psi_\nu(x) = \frac{\sin\big(\frac{2\nu}{ab}\mathrm{arctg}(e^{bx})\big)}{\cosh^{1/2}(bx)}. \tag{47}$$

C. Kasner-like case

In the case (2a), the conditions $\mathrm{O}_1\mathrm{O}_2 + \mathrm{O}_1\mathrm{O}_3 + \mathrm{O}_2\mathrm{O}_3 > 0$ with $\mathrm{O}_1 < 0, \mathrm{O}_2 > 0, \mathrm{O}_3 > 0$ are satisfied in the following domains[4] for O_k

$$\mathrm{O}_1 \in (-d_1, 0), \quad \mathrm{O}_2 \in (d_2, \infty), \quad \mathrm{O}_3 \in (d_3, \infty), \tag{48}$$

where $d_2 > d_1$, and where $d_3 = d_1 d_2/(d_2 - d_1)$ so $d_3 > d_1$. The full phase space sector of the Kasner-like evolution is defined as the union

$$\bigcup_{0<d_1<d_2} (-d_1, 0) \times (d_2, \infty) \times (d_3, \infty) \tag{49}$$

In the case of O_2 and O_3, the restrictions for domains (48) translate into the restrictions for the corresponding domains for the observables w_2 and w_3, due to (29), and read

$$w_2 \in (D_2, \infty), \quad w_3 \in (D_3, \infty), \tag{50}$$

where $D_2 = \kappa\gamma\lambda d_2$ and $D_3 = \kappa\gamma\lambda d_3$. Thus, quantization of the w_2 and w_3 observables can be done by analogy to the Kasner-unlike case. The spectra of the operators $\hat{w}_2$ and $\hat{w}_3$ are almost the same as the spectrum defined by (46) with the only difference that now $\nu > D_2$ and $\nu > D_3$, respectively[5].

The case of w_1 requires special treatment. Let us redefine the elementary observables corresponding to the 1-st direction as follows

$$\Omega_1 := -\frac{O_1}{b\cosh(b\mathcal{A}_1)}, \qquad \Omega_2 := \sinh(b\mathcal{A}_1). \tag{51}$$

The transformation (51) is canonical, since $\{\Omega_1, \Omega_2\} = 1$, and invertible. The domains transform as follows

$$O_1 \in (-d_1, 0), \quad \mathcal{A}_1 \in \mathbb{R} \qquad \longrightarrow \qquad \Omega_1 \in (0, d_1/b) =: (0, D_1), \quad \Omega_2 \in \mathbb{R}. \tag{52}$$

The observable v_1 in terms of redefined observables reads

$$v_1 = \frac{ab}{\hbar}\,\Omega_1\,(1 + \Omega_2^2), \qquad v_1 \in (0, \infty), \tag{53}$$

where $ab/\hbar = 12\pi G\gamma\lambda$. To quantize observables Ω_1 and Ω_2 we use the Schrödinger representation

$$\Omega_2 \to \hat{\Omega}_2 f(x) := -i\hbar\partial_x f(x), \qquad \Omega_1 \to \hat{\Omega}_1 f(x) := x f(x), \qquad f \in L^2(0, D_1). \tag{54}$$

Let us find an explicit form for the operator $\frac{ab}{\hbar}(\widehat{\Omega}_1 + \widehat{\Omega_1\Omega_2^2})$, corresponding to (53). Since $\Omega_1 > 0$, the following classical equality holds

$$\Omega_1\Omega_2^2 = \Omega_1^k \cdot \Omega_2 \cdot \Omega_1^{1-k-m} \cdot \Omega_2 \cdot \Omega_1^m, \tag{55}$$

---

[4] The case (2b) can be done by analogy.

[5] Spectra are insensitive to unitary transformations.

where $m, k \in \mathbb{R}$. This may lead to many operator orderings at the quantum level. This issue is further discussed in the appendix B and in the conclusion section.

We propose the following mapping (we set $\hbar = 1$)

$$\Omega_1\Omega_2^2 \to \widehat{\Omega_1\Omega_2^2} := \frac{1}{2}\bigg(\hat{\Omega}_1^k\hat{\Omega}_2\,\hat{\Omega}_1^{1-k-m}\hat{\Omega}_2\,\hat{\Omega}_1^m + \hat{\Omega}_1^m\hat{\Omega}_2\,\hat{\Omega}_1^{1-k-m}\hat{\Omega}_2\,\hat{\Omega}_1^k\bigg) = -x\partial^2_{xx} - \partial_x + mkx^{-1}, \tag{56}$$

which formally ensures the symmetricity of $\widehat{\Omega_1\Omega_2^2}$. The second equality in (56) may be verified via direct calculations.

Now, we define the following unitary transformation W

$$L^2([0, D_1], dx) \ni f(x) \mapsto Wf(x) := \sqrt{\frac{y}{2}}f\left(\frac{y^2}{4}\right) \in L^2([0, 2\sqrt{D_1}], dy). \tag{57}$$

One may verify that we have

$$W\partial_x W^\dagger = \frac{2}{y}\partial_y - \frac{1}{y^2}, \qquad W\partial^2_{xx}W^\dagger = \frac{4}{y^2}\partial^2_{yy} - \frac{8}{y^2}\partial_y + \frac{5}{y^4}\ . \tag{58}$$

Thus, the operator W transforms (56) into

$$-\partial^2_{yy} + \frac{1}{y^2}\bigg(4mk - \frac{1}{4}\bigg). \tag{59}$$

The eigenvalue problem for $\widehat{\Omega}_1 + \widehat{\Omega_1\Omega_2^2}$ reads

$$\bigg(-\partial^2_{yy} + \frac{1}{y^2}\bigg(4mk - \frac{1}{4}\bigg) + \frac{y^2}{4}\bigg)\Phi = \nu\,\Phi. \tag{60}$$

Now, we can see an advantage of the chosen ordering prescription (56). It enables finding a very simple form of the volume operator. Taking $k = m = 1/4$ turns (60) into

$$\bigg(-\partial^2_{yy} + \frac{y^2}{4} - \nu\bigg)\Phi = 0. \tag{61}$$

The problem is mathematically equivalent to the one dimensional harmonic oscillator in a 'box' with an edge equal to $2\sqrt{D_1}$. There are two independent solutions for a given ν

$$\Phi_{\nu,1} = N_1 e^{-y^2/4}\ {}_1F_1\bigg(-\frac{1}{2}\nu + \frac{1}{4}, \frac{1}{2}, \frac{y^2}{2}\bigg), \tag{62}$$

$$\Phi_{\nu,2} = N_2 y e^{-y^2/4}\ {}_1F_1\bigg(-\frac{1}{2}\nu + \frac{3}{4}, \frac{3}{2}, \frac{y^2}{2}\bigg), \tag{63}$$

where ${}_1F_1$ is a hypergeometric confluent function, $\Phi_{\nu,1}$ and $\Phi_{\nu,2}$ are even and odd cylindrical functions, respectively. A standard condition for the symmetricity of the operator defining the eigenvalue problem (61) leads to the vanishing of the wave functions at the boundaries (as the box defines the entire size of the 1-st direction). The solution (after retrieving of $\hbar$ and ab) reads[6].

$$\Phi = Nye^{-\frac{y^2}{4\hbar}}\ {}_1F_1\bigg(-\frac{1}{2}\frac{\nu}{ab} + \frac{3}{4}, \frac{3}{2}, \frac{y^2}{2\hbar}\bigg). \tag{64}$$

[6] We ignore the solution $\Phi_{\nu,1}$ because it cannot vanish at $y = 0$.

The solution (64) vanishes at $y = 0$ as Φ is an odd function. The requirement of vanishing at $y = 2\sqrt{D_1}$ leads to the equation

$$ {}_1F_1\bigg(-\frac{1}{2}\frac{\nu}{ab\hbar}+\frac{3}{4},\frac{3}{2},\frac{2D_1}{\hbar}\bigg)=0. \tag{65}$$

An explicit form of (65) reads

$$\sum_{n=0}^{\infty}\frac{\big(-\frac{1}{2}\frac{\nu}{ab}+\frac{3}{4}\big)_n}{\big(\frac{3}{2}\big)_n}\bigg(\frac{2D_1}{\hbar}\bigg)^n=0, \tag{66}$$

where $(a)_n = a(a+1)\dots(a+n-1)$. It results from (66) that the eigenvalues must satisfy the condition: $\nu \geq ab$.

D. Quantum volume operator

Classically we have

$$V = |w_1 w_2 w_3|^{1/3}. \tag{67}$$

One may verify that v_k Poisson commute and $\hat{v}_k$ commute, so we can take

$$\widehat{V}^3 := \hat{v}_1\hat{v}_2\hat{v}_3 = |\hat{w}_1\hat{w}_2\hat{w}_3|. \tag{68}$$

The eigenfunctions of the operator $\hat{w}_1\hat{w}_2\hat{w}_3$ have the form $F^{\lambda_1,\lambda_2,\lambda_3} := {f_1}^{\lambda_1}(x_1){f_2}^{\lambda_2}(x_2){f_3}^{\lambda_3}(x_3)$, where ${f_i}^{\lambda_i}(x_i)$ is an eigenvector of $\hat{w}_i$ with eigenvalue λ_i. The closure of the span of $F^{\lambda_1,\lambda_2,\lambda_3}$ is a Hilbert space, in which $\widehat{V}^3$ is a self-adjoint operator (by construction).

Due to the spectral theorem on self-adjoint operators [19], we have

$$V = (V^3)^{1/3} \quad\longrightarrow\quad \widehat{V}F^{\lambda_1,\lambda_2,\lambda_3} := \square\, F^{\lambda_1,\lambda_2,\lambda_3}, \tag{69}$$

where

$$\square := |\lambda_1\lambda_2\lambda_3|^{1/3}. \tag{70}$$

1. Kasner-unlike case

In the Kasner-unlike case we use the formula (46) to get

$$\square = |n_1 n_2 n_3|^{1/3}\, ab, \qquad n_1, n_2, n_3 \in 1, 2, 3, \dots, \tag{71}$$

which shows that the spectrum of the volume operator does not have equally distant levels. The volume $\square$ equal to *zero* is not in the spectrum. There exist a *quantum* of the volume which equals $\triangle := ab = 12\pi G\gamma\lambda\hbar$, and which defines the lowest value in the spectrum.

2. Kasner-like case

The spectrum in this case reads

$$\square := \bigcup_{0<d_1<d_2} \square_{d_1,d_2}, \qquad \square_{d_1,d_2} := \{\lambda_{d_1}\lambda_{d_2}\lambda_{d_3} \,|\, d_3 = d_1 d_2/(d_2-d_1)\}, \tag{72}$$

where λ_{d_1} is any value subject to the condition (66), $\lambda_{d_2} > D_2$ and $\lambda_{d_3} > D_3$ are given by (46). The volume $\square$ equal to *zero* is not in the spectrum.

E. Evolution

In this section we ignore the restrictions concerning the domains of O_1, O_2 and O_3, and we assume that the Hilbert space of the system is $L^2(\mathbb{R}^3, dxdydz)$. An inclusion of the restrictions would complicate the calculations without bringing any qualitative change into the picture of evolution.

The generator of evolution determined in (34) may be formally quantized, due to (35), as follows

$$\mathbb{H} \mapsto \hat{\mathbb{H}} = -\hbar^2(\partial_y\partial_z + \partial_x\partial_z + \partial_x\partial_y). \tag{73}$$

Since it is self-adjoint in $L^2(\mathbb{R}^3, dxdydz)$, a quantum evolution can be defined by an unitary operator

$$U = e^{-i\hbar\tau(\partial_y\partial_z+\partial_x\partial_z+\partial_x\partial_y)}, \qquad \tau \in \mathbb{R}. \tag{74}$$

Let us study an evolution of the expectation value of the directional volume $\hat{v}_1$

$$\langle\psi|U^{-1}\hat{v}_1U|\psi\rangle \tag{75}$$

Since $\hat{v}_1$ does not depend on y and z, we simplify our considerations by taking

$$U_1 = e^{-i\hbar\tau(\partial_z+\partial_y)\partial_x}. \tag{76}$$

If we are interested in the action of U_1 on the functions $f(x) \in L^2(\mathbb{R}, dx)$, then the derivatives $-i\frac{d}{dy}$ and $-i\frac{d}{dz}$ occurring in U_1 commute and, being self-adjoint, lead finally to real numbers. Let us call them k_y and k_z, respectively, and let us introduce the parameter $k = k_y + k_z$. Hence, U_1 further simplifies and reads

$$U_1 = e^{k\hbar\tau\partial_x}. \tag{77}$$

The action of U_1 on $f(x)$ reads

$$U_1f(x) = f(x + k\hbar\tau). \tag{78}$$

We recall that under the unitary mapping $L^2(\mathbb{R}, dx) \mapsto L^2(\mathbb{I}, dy)$, defined by (38), the operator $\hat{v}_1$ becomes $-ia\frac{d}{dy}$ on $L^2(\mathbb{I}, dy)$. Now, let us study an action of operator U_1 on the functions $\varphi(y) \in L^2(\mathbb{I}, dy)$. Straightforward calculation leads to

$$L^2(\mathbb{I}, y) \ni \varphi(y) \mapsto \frac{\varphi(\frac{2}{b}\arctan(e^{bx}))}{\cosh^{1/2}(bx)} \in L^2(\mathbb{R}, x), \tag{79}$$

and we have

$$U_1\frac{\varphi(\frac{2}{b}\arctan(e^{bx}))}{\cosh^{1/2}(bx)} = \frac{\varphi(\frac{2}{b}\arctan(e^{bx+bk\hbar\tau}))}{\cosh^{1/2}(bx+bk\hbar\tau)} \tag{80}$$

The transformation $\tilde{U}^{-1}$ gives

$$\frac{\varphi(\frac{2}{b}\arctan(e^{bx+bk\hbar\tau}))}{\cosh^{1/2}(bx+bk\hbar\tau)} \mapsto \frac{\varphi(\frac{2}{b}\arctan(e^{bk\hbar\tau}\tan(\frac{by}{2})))}{\sqrt{\frac{1}{2}\sin(by)(\tan(\frac{by}{2})e^{bk\hbar\tau} + \cot(\frac{by}{2})e^{-bk\hbar\tau})}} =: \varphi_\tau(y), \tag{81}$$

where $\varphi_{\tau=0}(y) = \varphi(y)$. Now, we observe that the symmetricity condition

$$\langle\varphi_\tau(y)|\hat{v}_1\varphi_\tau(y)\rangle = \langle\hat{v}_1\varphi_\tau(y)|\varphi_\tau(y)\rangle \tag{82}$$

leads to

$$\overline{\varphi}_\tau(\frac{\pi}{b})\varphi_\tau(\frac{\pi}{b}) - \overline{\varphi}_\tau(0)\varphi_\tau(0) = 0. \tag{83}$$

We use the result (81) to calculate the limits

$$\lim_{y\longrightarrow 0}\varphi_\tau(y) = e^{\frac{bk\hbar\tau}{2}}\varphi_0(0), \qquad \lim_{y\longrightarrow \frac{\pi}{b}}\varphi_\tau(y) = e^{-\frac{bk\hbar\tau}{2}}\varphi_0(\frac{\pi}{b}), \tag{84}$$

which turns (83) into

$$\overline{\varphi}_0(\frac{\pi}{b})\varphi_0(\frac{\pi}{b})e^{-bk\hbar\tau} - \overline{\varphi}_0(0)\varphi_0(0)e^{bk\hbar\tau} = 0. \tag{85}$$

It is clear that (85) can be satisfied $\forall\tau$ *iff* $\varphi_0(\frac{\pi}{b}) = 0 = \varphi_0(0)$. States with such a property belong to the domain of $\breve{w}$ defined by (46).

In order to construct the 'evolving states' that vanish at the boundaries, consider the basis vectors $f_n(y) = e^{i2bny}$. Then, $f_n(y) - f_m(y)$ satisfy the condition (85). Making use of (81) we get

$$f_n(y,\tau) = \left(\frac{i - e^{bk\hbar\tau}\tan(\frac{by}{2})}{i + e^{bk\hbar\tau}\tan(\frac{by}{2})}\right)^{2n}\sqrt{\frac{1+\tan^2(\frac{by}{2})}{e^{-bk\hbar\tau} + e^{bk\hbar\tau}\tan^2(\frac{by}{2})}}, \tag{86}$$

where $f_n(y,\tau) := f_{n,\tau}(y)$. Moreover we have

$$-ia\frac{d}{dy}f_n(y,\tau) = -i\frac{ab}{2}(1+\tan^2(\frac{by}{2}))f_n(y,\tau)\frac{1}{1+e^{2bk\hbar\tau}\tan^2(\frac{by}{2})}\left(\frac{(1-e^{2bk\hbar\tau})\tan(\frac{by}{2})}{1+\tan^2(\frac{by}{2})} + i4ne^{bk\hbar\tau}\right). \tag{87}$$

Using the substitution $x = \tan(\frac{by}{2})$ we get

$$\begin{aligned} &\langle f_m| - ia\frac{d}{dy}f_n\rangle = \\ &-ia\int_0^\infty \left(\frac{i - e^{bk\hbar\tau}x}{i + e^{bk\hbar\tau}x}\right)^{2(n-m)} \frac{(e^{-bk\hbar\tau} - e^{bk\hbar\tau})x}{(e^{-bk\hbar\tau} + e^{bk\hbar\tau}x^2)^2}\, dx \\ &+4an\int_0^\infty \left(\frac{i - e^{bk\hbar\tau}x}{i + e^{bk\hbar\tau}x}\right)^{2(n-m)} \frac{1+x^2}{(e^{-bk\hbar\tau} + e^{bk\hbar\tau}x^2)^2}\, dx. \end{aligned} \tag{88}$$

Another substitution $z = e^{bk\hbar\tau}x$ leads to

$$\begin{aligned} &\langle f_m| - ia\frac{d}{dy}f_n\rangle = \\ &-ia(e^{-bk\hbar\tau} - e^{bk\hbar\tau})\int_0^\infty \left(\frac{i - z}{i + z}\right)^{2(n-m)} \frac{z}{(1+z^2)^2}\, dz \\ &+4an\int_0^\infty \left(\frac{i - z}{i + z}\right)^{2(n-m)} \frac{e^{bk\hbar\tau} + e^{-bk\hbar\tau}z^2}{(1+z^2)^2}\, dz \end{aligned} \tag{89}$$

Finally, we obtain

$$\langle f_m| - ia\frac{d}{dy}f_n\rangle = \begin{cases} \frac{ia}{4(n-m)^2-1}(1 - 8n(n-m))\sinh(bk\hbar\tau), & n \neq m \\ ia\sinh(bk\hbar\tau) + 2\pi na\cosh(bk\hbar\tau), & n = m. \end{cases} \tag{90}$$

Now, let us introduce $g_{nm}(y,\tau) := \frac{f_n(y,\tau)-f_m(y,\tau)}{\sqrt{\frac{2\pi}{b}}}$ so that $\|g_{nm}\| = 1$. One has

$$\langle g_{nm}| - ia\frac{d}{dy}g_{nm}\rangle = (n+m)ab\cosh(bk\hbar\tau) = \frac{n+m}{2}\Delta\cosh(bk\hbar\tau). \tag{91}$$

The expectation value of the operator (91), defining the volume operator, is similar to the classical form (29). The vectors g_{nm} may be used in the construction of a basis of the space of states such that $\varphi_0(\frac{\pi}{b}) = 0 = \varphi_0(0)$.

IV. SUMMARY AND CONCLUSIONS

Turning the Big Bang into the Big Bounce in the Bianchi I universe is due to the modification of the model at the *classical* level by making use of the loop geometry (in complete analogy to the FRW case). The modification is parameterized by a continuous parameter λ to be determined from observational astro-cosmo data.

The reduced phase space of our system is higher dimensional with nontrivial boundaries. This requires introducing a few new elements, comparing to the FRW case, into our method: 1. Our parameter does *not* need to coincide with the scalar field, commonly used in loop quantum cosmology, and it simplifies the form of a volume operator. 2. We introduce the so-called *true* Hamiltonian. It generates a flow in the family of volume quantities, enumerated by the evolution parameter. It proves an independence of the spectrum of the volume operator on the evolution. 3. We divide the phase space of the system into two distinct regions: Kasner-like and Kasner-unlike. 4. We identify domains, spectra and eigenvectors of *self-adjoint* directional volumes and total volume operators in the Kasner-unlike case. 5. We identify the *peculiarity* of the Kasner-like case due to complicated boundary of the phase space region. We propose to overcome this problem by dividing this region further into smaller regions, but with simpler boundaries. 6. Given a small subregion for Kasner-like case, we propose a canonical *redefinition* of phase space coordinates in such a way, that we can arrive at relatively simple form of volume operator and at the same time can simply encode the boundary of the region into the Schrödinger representation. Then, from a number of different operator orderings we chose the simplest one. We find domain, spectrum and eigenvectors of the volume operator. The spectrum is given in an implicit form in terms of special functions. 7. Having the true (self-adjoint) Hamiltonian, we introduce an *unitary* operator with an evolution parameter.

The spectrum of the volume operator, parameterized by λ, is bounded from below and *discrete*. An evolution of the expectation value of the volume operator is similar to the classical case. We have presented the evolution of only a single directional volume operator. One may try to generalize this procedure to the *total* volume operator. Analyzes, in the case of the Kasner-like dynamics, are complicated and will be presented elsewhere.

Discreteness of space at the quantum level may lead to a *foamy* structure of spacetime at the semi-classical level. A possible astro-cosmo observation of *dispersion* of photons travelling over cosmological distances across the Universe might be used to determine the value of otherwise free parameter λ. The discreteness is also specific to the FRW case [9]. The difference is that in the Bianchi I case the variety of possible quanta of a volume is much richer. On the other hand, the Bianchi type cosmology seems to be more realistic than the FRW case, near the cosmological singularity. Thus, an expected foamy structure of space

may better fit cosmological data. Various forms of discreteness of space may underly many approaches in fundamental physics. So its examination may be valuable.

Quantum cosmology calculations are plagued by quantization *ambiguities*. In particular, there exists a huge freedom in ordering of elementary operators defining compound observables, which may lead to different quantum operators. Classical commutativity of variables does not extend to corresponding quantum operators. Such ambiguities can be largely reduced after some quantum astro-cosmo data become available. Confrontation of theoretical predictions against these data would enable finding realistic quantum cosmology models.

Our nonstandard loop quantum cosmology method, successfully applied so far mainly to the FRW type models, seems to be highly efficient and deserves further development and application to sophisticated cosmological models of general relativity.

Appendix A: Unitarily non-equivalent volume operators

In both Kasner-like and Kasner-unlike cases, we have reduced the Hilbert space by removing the double *degeneracy* of eigenvalues for the volume operators (see the discussion after equations (44) and (61)). We have used the 'natural' condition that the wave function should vanish at the boundaries of an interval. However, there are also other mathematically well-defined choices for the boundary conditions. We will demonstrate this non-uniqueness for the Kasner-unlike case. Similar reasoning applies to another case.

Let us begin with the equation (44)

$$-a^2\frac{d^2}{dy^2}f=\nu^2 f, \qquad y\in(0,\pi/b), \tag{A1}$$

which has the solution

$$f_\nu = N_1\sin(\frac{\nu}{a}y)+N_2\cos(\frac{\nu}{a}y), \qquad N_1,N_2\in\mathbb{C}, \tag{A2}$$

for each value of $\nu\in\mathbb{R}_+$ ($\nu\mapsto-\nu$ does not produce any new space of solutions). Our task is the determination of self-adjointness of $\breve{w}:=\sqrt{-a^2\frac{d^2}{dy^2}}$ and removing the double degeneracy of eigenvalues. The symmetricity condition reads

$$\int_I \bar{f}f''=\bar{f}f'\Big|_0^{\pi/b}-\bar{f}'f\Big|_0^{\pi/b}+\int_I\bar{f}''f. \tag{A3}$$

We can set:

- $f(0)=f(\pi/b)=0\Rightarrow f_\nu=\sin(\frac{\nu}{a}y), \quad \nu=ab,2ab,3ab,\ldots$
- $f'(0)=f'(\pi/b)=0\Rightarrow f_\nu=\cos(\frac{\nu}{a}y), \quad \nu=0,ab,2ab,3ab,\ldots$
- $f(0)=f'(\pi/b)=0\Rightarrow f_\nu=\sin(\frac{\nu}{a}y), \quad \nu=\frac{1}{2}ab,\frac{3}{2}ab,\frac{5}{2}ab,\ldots$

where $ab=12\pi\hbar G\gamma\lambda$. All these choices are non-equivalent, since they lead to different spectra.

Appendix B: Standard quantization

Let us change the coordinates of the Kasner-like sector phase space (Ω_1, Ω_2), defined by (51), into a new canonical pair as follows

$$X := \sqrt{2\Omega_1} \quad \text{and} \quad P := \Omega_2 \sqrt{2\Omega_1}, \tag{B1}$$

where

$$(X, P) \in (0, \sqrt{2d_1/b}) \times \mathbb{R}, \qquad \{X, P\} = 1. \tag{B2}$$

In the new variables the volume (53) reads

$$\frac{1}{4\pi G\gamma\lambda}\, v_1 = \frac{1}{2}P^2 + \frac{1}{2}X^2. \tag{B3}$$

Thus, in these variables the volume has a form of the Hamiltonian of the harmonic oscillator in a 'box' $(0, \sqrt{2d_1/b})$.

In the Schrödinger representation, i.e. $\hat{X} := x$ and $\hat{P} := -i\hbar\partial_x$, a standard quantization yields

$$\frac{1}{4\pi G\gamma\lambda}\, \hat{v} = -\frac{\hbar^2}{2}\partial^2_{xx} + \frac{1}{2}x^2, \tag{B4}$$

which corresponds to the 'nonstandard' quantization (60) with the parameters $m = k = 1/4$ and $y = \sqrt{2}x$ (with $\hbar = 1$).

Thus, we can see that the prescription defined by (55) and (56) includes not only a standard prescription, but many others. We have completed only one, corresponding to the well known harmonic oscillator, as an illustration.

---

[1] V. A. Belinskii, I. M. Khalatnikov and E. M. Lifshitz, Adv. Phys. **19** (1970) 525; **31** (1982) 639.

[2] V. A. Belinskii, I. M. Khalatnikov and M. P. Ryan 'The oscillatory regime near the singularity in Bianchi type IX universes', Preprint of Landau Institute for Theoretical Physics, Moscow 1971; M. P. Ryan, Annals of Physics, **70** (1972) 301.

[3] G. Montani, M. V. Battisti, R. Benini and G. Imponente, "Classical and Quantum Features of the Mixmaster Singularity," Int. J. Mod. Phys. A **23**, 2353 (2008) [arXiv:0712.3008 [gr-qc]].

[4] T. Damour, M. Henneaux and H. Nicolai, "Cosmological billiards," Class. Quant. Grav. **20**, R145 (2003) [arXiv:hep-th/0212256].

[5] P. Dzierzak and W. Piechocki, "Bianchi I model in terms of nonstandard loop quantum cosmology: Classical dynamics, " Phys. Rev. D **80**, 124033 (2009) [arXiv:0909.4211 [gr-qc]].

[6] P. Dzierzak, J. Jezierski, P. Malkiewicz and W. Piechocki, "The minimum length problem of loop quantum cosmology," Acta Phys. Polon. B **41** (2010) 717 [arXiv:0810.3172 [gr-qc]].

[7] P. Dzierzak, P. Malkiewicz and W. Piechocki, "Turning big bang into big bounce: I. Classical dynamics," Phys. Rev. D **80**, 104001 (2009) [arXiv:0907.3436].

[8] P. Malkiewicz and W. Piechocki, "Energy Scale of the Big Bounce," Phys. Rev. D **80** (2009) 063506 [arXiv:0903.4352 [gr-qc]].

[9] P. Malkiewicz and W. Piechocki, "Turning big bang into big bounce: II. Quantum dynamics," Class. Quant. Grav. **27** (2010) 225018 [arXiv:0908.4029 [gr-qc]]

[10] J. Mielczarek and W. Piechocki, “Observables for FRW model with cosmological constant in the framework of loop cosmology,” Phys. Rev. D **82** (2010) 043529 [arXiv:1001.3999 [gr-qc]].
[11] M. Bojowald, “Homogeneous loop quantum cosmology,” Class. Quant. Grav. **20** (2003) 2595 [arXiv:gr-qc/0303073].
[12] D. W. Chiou, “Loop Quantum Cosmology in Bianchi Type I Models: Analytical Investigation,” Phys. Rev. D **75** (2007) 024029 [arXiv:gr-qc/0609029].
[13] D. W. Chiou, “Effective Dynamics, Big Bounces and Scaling Symmetry in Bianchi Type I Loop Quantum Cosmology,” Phys. Rev. D **76** (2007) 124037 [arXiv:0710.0416 [gr-qc]].
[14] L. Szulc, “Loop quantum cosmology of diagonal Bianchi type I model: Simplifications and scaling problems,” Phys. Rev. D **78** (2008) 064035 [arXiv:0803.3559 [gr-qc]].
[15] M. Martin-Benito, G. A. M. Marugan and T. Pawlowski, “Physical evolution in Loop Quantum Cosmology: The example of vacuum Bianchi I,” Phys. Rev. D **80** (2009) 084038 [arXiv:0906.3751 [gr-qc]].
[16] A. Ashtekar and E. Wilson-Ewing, “Loop quantum cosmology of Bianchi I models,” Phys. Rev. D **79** (2009) 083535 [arXiv:0903.3397 [gr-qc]].
[17] M. Bojowald, D. Cartin and G. Khanna, “Lattice refining loop quantum cosmology, anisotropic models and stability,” Phys. Rev. D **76** (2007) 064018 [arXiv:0704.1137 [gr-qc]].
[18] M. Geiller and W. Piechocki, in progress.
[19] M. Reed and B. Simon, *Methods of Modern Mathematical Physics* (Academic Press, San Diego, 1975).